Classification of finite groups of order ≤ 200 in terms of solvability

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Abstract:

The study of solvable groups has been, and continues to be, the most significant area of work in group theory. Our aim in this work is to classify groups of orderup to 200 in terms of solvability. Our starting point of this classification is to prove that any group with fewer than 60 elements is solvable. Finally, we will show that any group of order up to 200 and not 60, 120, 168, 180 is solvable.

Preliminaries:

The following results will play an important role in our discussion. We begin by recalling that if $|G| = p^n m$ where $(p, m) = 1$, then the subgroup of order p^n is called $p-$Sylowsubgroup of G. The Sylow Theorems were originally published in 1872 by P. L. M, Sylow (1832-1919). These theorems give information about a subgroups offinite groups (provide a partial converse to the Lagrange's theorem).

Theorem1.1:(First Sylow Theorem) Let Gbe a finite group, if $p$is prime and $p/|G|$;then there exists a $p-$Sylow subgroup of G.

Theorem 1.2:(Second Sylow Theorem) Let $|G| = p^n q$,where $p$is prime and $(p, q) = 1$,if P is a $p-$ Sylow subgroup of G and H is any subgroup of order a power of p, then $H \leq xPx^{-1}$ for some $x \in G$. In particular, any two $p-$ Sylow subgroups of G are conjugate.

Corollary 1.3: A $p-$Sylow subgroup P of a group$G$ is normal if and only if$P$ is unique $p-$ Sylow.

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Theorem 1.4 (Third Sylow Theorem):The number of $p-$ Sylow subgroups of the group$G$ divides $|G|$ and is of the form $1+kp$ for some nonnegative integer k.Generally in group theory when we want information about a group we break it up into smaller subgroups and their quotients and study their properties. It is time to give the formal definition of a solvable group.

Definition 1.5: A group G is solvable if it has a subnormal series $G=G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n=\{e\}$, where each factor$G_i/G_{i+1}$is abelian.

Remark: Any abelian group is solvable. This means that non solvable groups are non abelian groups. Converse of the last remark is not true, as the S_4 shows.

Theorem 1.6: Let Gbe a solvable group; then the following hold
(1) Every subgroup of G is solvable.
(2) If $N \trianglelefteq G$,then ${}^{G}/_{N}$is solvable.

Theorem 1.7: Let G be a group,$N \trianglelefteq G$such that both $N$and ${}^{G}/_{N}$ are solvable. Then $G$is solvable.

Corollary 1.8: The product of two normal solvablesubgroupsis solvable. Towards achieving our goal, we need the following theorem.

Theorem 1.9 [5]: Every group of odd order is solvable.

Theorem 1.9 is due to W-Feit and J. Thompson where the proof is appeared in Pacific Jr. Math. 13, pages 775-1029, and this issue contains only this result with its proof.

Lemma 1.10 [1]: If a group Ghas a subgroup H such that $|G| \nmid [G:H]!$, than H contains a non-trivial normal sub group of G. One of the first great results in twentieth- century group theory isthe Burnside's Theorem, which is stated as following.

Theorem 1.11: Any group of order $p^m q^n$ (where $p,q$are prime) and m,n are nonnegative integers, is solvable.

Proposition 1.12 [4]: Let Gbe a group of order $2m$,where $m$is odd. Then G is solvable.

Proposition 1.13 [4]: If a finite group $G$is not solvable, than$4/|G|$
Corollary 1.14: Let G be a finite group whose order is the product of distinct (non-repeated) primes. Then G is solvable.
Definition1.15[3]: The general linear group $GL(2,p)$consists of all invertible 2×2 matrices over $Z_p = \{0,1,2 \ldots p-1\}$.In case of all such matrices with unite determinant is called special linear group and denoted by $SL(2,p)$. To calculate the order of the group$SL(2,p)$,we count the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}: (a,b,c,d \in Z_p$ and $ad - bc = 1)$. We consider two cases: 1- If $c = 0$,then there are $p(p-1)$choices for a,b,dwhich make $ad - bc = 1$(since $a,b$are arbitrary, except that $a \neq 0$and $d$is determined by a). 2 If $c \neq 0$,there are $p^2(p-1)$choices for a,b,c,dsuch that $ad - bc = 1$ (sincea,d may be chosen arbitrarily and $c$is any non-zero element of Z_p and b is determined) Hence $|SL(2,p)| = p(p-1) + p^2(p-1) = p(p^2-1)$.
If $p = 7$, then $|SL(2,7)| = 7(48) = 336$, we also notice that $Z(SL(2,7)) = \{I, -I\}$. Now, the group $SL(2,7)/Z(SL(2,7)$ is a group of order $\frac{336}{2} = 168$, and this group is denoted by $PSL(2,7)$ (projective special linear group).
proposition1.16[3]. The group $PSL(2,7)$ is non-abelian simple group.

Main Results

Theorem 2.1: Any group G of order n with $n < 60$ is solvable.**Proof :**Consider any positive integer n with $n < 60$ hence $n < 2^2.3.5$. Then n cannot be written as product of four primes. Thus $n$has at most three distinct prime factors. Hence if $1 \leq n < 60, n \neq 2.3.5$ and $2.3.7$, then is of type $p^n q^m$where p,q are primes and m,n non-negative integers. Thus by Theorem (1.11) and by corollary (1.14) any group of order < 60 is solvable.
Theorem 2.2: Any group of order up to 200 and not 60, 120, 168 and 180 is solvable.
Proof: We need to consider the following cases
1- $60 < |G| \leq 84$.

Any integer between 60 and 84 is either odd or twice of odd or of the form $p^n q^m$ or product of three distinct primes. Hence any group of order k where
$60 < k < 84$, is solvable.
Now, we show any group of order 84 is solvable, since $|G| = 2^2.3.7$. By Theorem (1.1) G has a subgroup oforder 7, and the number of such subgroups is congruent to 1 mod 7 and dividing $|G|$. Since 1,8,15,…71 are only positive integer ≤ 84 that are congruent to 1 mod 7 and among those only the number 1 divides $|G|$, hence G has only one subgroup of order 7 (say H), and by corollary 1.3 $H \trianglelefteq G, H$ is solvable (since H is cyclic) also $|G/H| = 12$ which is solvable by last theorem. So by theorem (1.7) G is solvable.
2- $84 < |G| < 120$. Any integer between 84 and 120 is either odd or twice of odd or of the form $p^m q^n$. Hence G is solvable.
3- $120 < |G| < 168$ By theorem 1.9, proposition1.12, corollary 1.14 and theorem 1.11 groups of order n where $120 < n < 168$ and $n \neq 132, 140, 156$ are solvable.
It remains only groups of order 132,140,156.First, $132 = 2^2.3.11$,and let N_{11}denote the number of Sylow 11-subgroups, N_3 denote the number of 3-Sylow subgroups and N_2 the number of 2-Sylow subgroups. By theorem 1.4,$N_{11} = 11k + 1$, for some integer $k \geq 0$ and N_{11} divides 132.Hence $N_{11} = 1$ or 12. If $N_{11} = 1$ then G has unique Sylow 11-subgroup which must be normal, then it is not hard to see that G is solvable .Suppose $N_{11} = 12$, this mean that G has 12 Sylow 11-subgroup of order 11, hence G contains 120 elements of order 11. On the other hand if $N_3 = 4$, then G has four Sylow 3-subgroup of order 3, hence G contains 8 elements of order 3. We notice that if $N_{11} = 12$ and $N_3 = 4$, then there is only room for one Sylow 2-subgroup of order 4 (sayH), this means that $H \trianglelefteq G$, and H is solvable, also G/H is a group of order 33, hence G/H is solvable and by (1.7)G is solvable.For the group$G$ of order $140 = 2^2.5.7$. By Theorem (1.4)such a group has only one subgroup H of order 7, hence its normal and solvable also the

group G/H is solvable, and by (1.7) G is solvable. Finally, by pursuing an analogous method as was done in the group of order 140, we can show that any group of order 156 is solvable.

4 $168 < |G| < 180$, $180 < |G| \leq 200$.As in the previous cases we can show that any group of order $n, 168 < n < 180$ and of order m where $180 < m \leq 200$ is solvable. And this completes the proof of the theorem.

Proposition 203: There exists a non- solvable groups of order 60, 120, 168 and 180.

Proof: (i) $|A_5| = 60$ and A_5 is non-abelian simple group, this means that $\{\rho_0\}$ is the only normal subgroup of A_5 implies that $A_5 > \{\rho_0\}$ is the only subnormalserise of A_5 but $A_5/\{\rho_0\} \cong A_5$ which is not abelian, hence A_5 is not solvable.

(ii)$|S_5| = 120$, is not solvable because if S_5 is solvable then A_5 is solvable, which is a contradiction. Hence S_5 is not solvable.

(iii) $|PSL(2,7)| = 168$, and by proposition (1.16) the group PSL(2,7) is non-abelian simple group, hence PSL(2,7) is not solvable.

(iv) $|A_5 \times \mathbb{Z}_3| = 180$, and $A_5 \times \mathbb{Z}_3$ is not solvable, because if $A_5 \times \mathbb{Z}_3$ is solvable then $A_5 \cong A_5 \times \{0\}$ is solvable, which is a contradiction. Hence $A_5 \times \mathbb{Z}_3$ is not solvable.

تصنيف الزمر المنتهية ذات الرتبة 200 ≥ بالنسبة للقابلية للحل

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المستخلص:

دراسة الزمر القابلة للحل كانت ومازالت من أهم موضوعات البحث في نظريات الزمر. الهدف الرئيسي في هده الورقة هو تصنيف الزمر المنتهية حتى رتبة 200 من حيت القابلية للحل، في البداية نبرهن أن كل زمرة رتبتها اقل من 60 تكون قابلة للحل وفي النهاية سوف نثبت أن كل زمرة رتبتها اقل من او تساوي 200عدا 60, 120 ,168, 180 تكون قابلة للحل.

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